Review on Week 6/7

Cluster Point

When talking about limit, we need to consider points that are "close" to each other.

Definition (c.f. Definition 4.1.1)**.** Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a *cluster point* of A if for every $\delta > 0$, there exists some $x \in A$ and $x \neq c$ such that $|x - c| < \delta$.

Remark. The point c may or may not be in A. Also, points in A may or may not be a cluster point. Observe the following examples:

Example (c.f. Example 4.1.3)**.** Let's visualize the sets and find their cluster points.

(a) The set of cluster point of $A_1 = (0, 1)$ is $[0, 1]$.

(b) The set of cluster point of $A_2 = \{0, 1\}$ is $\emptyset$.

(c) The set of cluster point of $A_3 = \mathbb{N}$ is $\emptyset$.

(d) The set of cluster point of $A_4 = \{1/n : n \in \mathbb{N}\}$ is $\{0\}$.

(e) The set of cluster point of $A_5 = \mathbb{Q}$ is $\mathbb{R}$.

Theorem (c.f. Theorem 4.1.2)**.** *Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. c is a cluster point of A if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.*

Limit of Function

The definition of limit of a function is similar to that of a sequence.

Definition (c.f. Definition 4.1.4)**.** Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. A real number L is said to be a *limit* of f at c, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x \in A$ and $0 < |x - c| < \delta$,

$$|f(x) - L| < \varepsilon.$$

In this case, f is said to *converge* to L at c and we denote

$$L = \lim_{x \to c} f(x).$$

Remark. We can only discuss the limit of a function at cluster points of its domain. For example, if f is a function defined on A_4 in the previous example, then we can only talk about the limit of f at 0. Also, if a function converges at a point, then the limit is unique.

Since we can formulate cluster point by sequence, limit of functions can also be formulated by sequence.

Theorem (c.f. Theorem 4.1.8)**.** *Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. Let $L \in \mathbb{R}$. The following are equivalent:*

(i) $\lim_{x \to c} f(x) = L$.

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

Divergence Criteria (c.f. 4.1.9). *Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function.*

(a) If $L \in \mathbb{R}$, then f does not have a limit L at c if and only of there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge to L.

(b) The function f does not have a limit at c if and only of there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge in $\mathbb{R}$.

Limit at Infinity

Definition (c.f. Definition 4.3.10). Let $A \subseteq \mathbb{R}$ with $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. $L \in \mathbb{R}$ is said to be a limit of f as $x \to \infty$ if for any $\varepsilon > 0$, there exists $K > a$ such that

$$|f(x) - L| < \varepsilon, \quad \forall x > K.$$

In this case, we write

$$\lim_{x \to \infty} f(x) = L.$$

Remark. Can you formulate the definition for the limit as $x \to -\infty$?

Theorem (c.f. Theorem 4.3.11). *Let $A \subseteq \mathbb{R}$ with $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. Let $L \in \mathbb{R}$. The following are equivalent:*

(i) $\lim_{x \to \infty} f(x) = L$.

(ii) For every sequence (x_n) in (a, ∞) that is properly divergent to ∞, the sequence $(f(x_n))$ converges to L.

Examples

Example 1. Establish the convergence of the following limits.

(a) $\lim_{x \to 10} x^2$. (b) $\lim_{x \to 2} \dfrac{x^3 - 4}{x^2 + 1}$. (c) $\lim_{x \to \infty} \dfrac{2x^2 + x + 1}{x^2 + 3}$.

Solution. We prove them by definition.

(a) Note that

$$|x^2 - 100| = |x + 10||x - 10|, \quad \forall x \in \mathbb{R}.$$

If $|x - 10| < 1$, then

$$|x + 10| \leq |x - 10| + 20 < 21.$$

Let $\varepsilon > 0$. Take $\delta = \min\{\varepsilon/21, 1\}$. Then whenever $0 < |x - 10| < \delta$,

$$|x^2 - 100| = |x + 10||x - 10| < 21\delta \leq \varepsilon.$$

Hence $\lim_{x \to 10} x^2 = 100$.

(b) Note that

$$\left|\frac{x^3-4}{x^2+1}-\frac{4}{5}\right| = \frac{|5x^3-4x^2-24|}{5(x^2+1)} = \frac{|5x^2+6x+12|}{5(x^2+1)}|x-2|, \quad \forall x \in \mathbb{R}.$$

If $|x-2|<2$, then $0<x<4$, so

$$\frac{|5x^2+6x+12|}{5(x^2+1)} < \frac{5(4)^2+6(4)+12}{5(0^2+1)} = \frac{116}{5}.$$

Let $\varepsilon>0$. Take $\delta = \min\{2, 5\varepsilon/116\}$. Then whenever $0<|x-2|<\delta$,

$$\left|\frac{x^3-4}{x^2+1}-\frac{4}{5}\right| < \frac{116}{5}\delta \le \varepsilon.$$

Hence $\lim_{x\to 2}\frac{x^3-4}{x^2+1} = \frac{4}{5}$.

(c) Note that

$$\left|\frac{2x^2+x+1}{x^2+3}-2\right| = \frac{|x-5|}{x^2+3} \le \frac{|x-5|}{(x-5)^2+10x-22}, \quad \forall x \in \mathbb{R}.$$

If $x > 22/10 = 11/5$, then

$$\left|\frac{2x^2+x+1}{x^2+3}-2\right| < \frac{|x-5|}{(x-5)^2+10x-22} < \frac{|x-5|}{(x-5)^2+0} = \frac{1}{x-5}.$$

Let $\varepsilon>0$. Take $K=\max\{11/5, 1/\varepsilon+5\}$. Then whenever $x>K$,

$$\left|\frac{2x^2+x+1}{x^2+3}-2\right| < \frac{1}{x-5} < \frac{1}{K-5} \le \varepsilon.$$

Hence $\lim_{x\to\infty}\frac{2x^2+x+1}{x^2+3} = 2$.

Example 2. Show that the following limit does not exist.

(a) $\lim_{x\to 0}\frac{1}{x}$.

(b) $\lim_{x\to\infty}\sin(x)$.

Solution. We can apply **Divergence Criteria**.

(a) Consider the sequence $(1/n)$. Note that $1/n \neq 0$ for all $n \in \mathbb{N}$ and $\lim 1/n = 0$. Also, the sequence $(1/(1/n)) = (n)$ is divergent. Hence the limit does not exist.

(b) Consider the sequence $(n\pi/2)$. Note that this sequence is properly divergent to ∞. Also, the sequence $\sin(n\pi/2) = (1, 0, -1, 0, 1, 0, ...)$ is divergent. Hence the limit does not exist.

Exercises

Question 1 (c.f. Section 4.1, Ex.10(b))**.** Use the definition of limit to show that

$$\lim_{x\to -1} \frac{x+5}{2x+3} = 4.$$

Solution. Note that

$$\left|\frac{x+5}{2x+3} - 4\right| = \frac{7}{|2x+3|}|x+1|, \quad \forall x \in \mathbb{R}.$$

If $|x+1| < 1/4$, then $-5/4 < x < -3/4$. Hence

$$\frac{1}{2} < 2x+3 < \frac{3}{2}.$$

Let $\varepsilon > 0$. Take $\delta = \min\{1/4, \varepsilon/14\}$. Then whenever $0 < |x+1| < \delta$,

$$\left|\frac{x+5}{2x+3} - 4\right| < \frac{7}{1/2}\delta \le \varepsilon.$$

The result follows.

Question 2 (c.f. Section 4.1, Ex.8)**.** Show that $\lim_{x\to c} \sqrt{x} = \sqrt{c}$ for any $c > 0$.

Solution. Note that

$$|\sqrt{x} - \sqrt{c}| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \le \frac{|x-c|}{\sqrt{c}}, \quad \forall x \ge 0.$$

Let $\varepsilon > 0$. Take $\delta = \varepsilon\sqrt{c}$. Then whenever $0 < |x-c| < \delta$ and $x \ge 0$,

$$|\sqrt{x} - \sqrt{c}| \le \frac{|x-c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} = \varepsilon.$$

Question 3 (c.f. Section 4.3, Ex.9)**.** Show that if $f : (a,\infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = 0$.

Solution. Note that

$$|f(x) - 0| = \frac{1}{x}|xf(x)| \le \frac{1}{x}(|xf(x) - L| + |L|), \quad \forall x > 0.$$

Since $\lim_{x\to\infty} xf(x) = L$, there exists $K_1 > a$ such that whenever $x > K_1$,

$$|xf(x) - L| < 1.$$

It follows that whenever $x > 0$ and $x > K_1$,

$$|f(x) - 0| \le \frac{1}{x}(1 + |L|).$$

Let $\varepsilon > 0$. Take

$$K = \max\left\{0, K_1, \frac{1+|L|}{\varepsilon}\right\}.$$

Then whenever $x > K$,

$$|f(x) - 0| \le \frac{1+|L|}{x} < \frac{1+|L|}{K} \le \varepsilon.$$

Question 4 (c.f. Section 4.1, Ex.14)**.** Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has limit L at 0, and let $a > 0$. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = f(ax)$ for $x \in \mathbb{R}$, show that $\lim_{x \to 0} g(x) = L$.

Solution. Let ε. Since $\lim_{x \to 0} f(x) = L$, there exists $\delta_1 > 0$ such that whenever $0 < |x| < \delta_1$,

$$|f(x) - L| < \varepsilon.$$

Take $\delta = a\delta_1$. Then whenever $0 < |x| < \delta$, we have $0 < |ax| < \delta_1$. Therefore

$$|g(x) - L| = |f(ax) - L| < \varepsilon.$$